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Optimal Material Selection Using Branch and Bound Techniques

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A methodology for structural synthesis is presented in which the optimal material selection for the truss structures is treated in terms of (0, 1) variables. Structural member sizes and material selection variables are treated simultaneously as design variables. Optimization is carried out by generating and solving a sequence of explicit approximate problems using a branch and bound strategy. Intermediate design variables and intermediate response quantities are used to enhance the quality of the approximate design problems. Numerical results for example problems are presented to illustrate the efficiency of the design procedure set forth.

Introduction

TRADITIONALLY, structural optimization problems consider only sizing and configuration variables. Several approaches have been developed for the efficient solution of these kinds of problems. The most effective techniques are based on approximation concepts, in which the solution of the original implicit optimization problem is replaced by the generation and solution of a sequence of explicit approximate design optimization problems.[1,2] The efficiency of the procedure is strongly dependent on the quality of the approximations used. High-quality approximations allow the use of larger move limits, which in turn reduces the number of real structural analyses required to find the optimal design.

In the present paper the inclusion of material selection for sizing problems is studied. That is, the problem consists of allocating a material from many candidate materials to each independent structural member while simultaneously determining sizing variables subject to various behavior requirements. This problem was originally studied in Ref. 3 and more recently using dual methods in Refs. 4 and 5. The objective function to be considered is the weight or the scaled weight of the structure such that each term represents the relative cost of each material. The scaling is important from a practical point of view, since the best material for a given structural member should be determined by both the cost and the weight of the structure. Choosing the optimal material represents an inherently combinatorial problem that lends itself to a formulation in terms of (0, 1) variables. Several techniques can be used to solve a (0, 1) continuous mixed design optimization problem (e.g., simulated annealing, neural networks, branch and bound techniques).

In this work the integrated sizing-material selection optimization problem is formulated as a nonlinear mathematical programming problem involving both continuous and (0, 1) variables. A strategy that combines approximation concepts and branch and bound techniques is the solution method adopted.

Problem Formulation

Only truss structures will be treated in the present paper. However, it is expected that the entire formulation and solution technique discussed in the following sections will be applicable to other important kinds of structures.

For a generalized truss design problem, the optimization problem can be stated as follows:

Minimize

$$W = \sum_{i=1}^{n} c_i \rho_i A_i L_i \tag{1a}$$

subject to

$$g_j(A_i, M_k) \leq 0 \qquad j = 1, \ldots, m \tag{1b}$$

$$\underline{A}_i \leq A_i \leq \bar{A}_i \qquad i = 1, \ldots, n \tag{1c}$$

$$M_k = \left(c_k, \rho_k, E_k, \sigma_k^L, \sigma_k^U\right) \qquad k = 1, \ldots, k_M \tag{1d}$$

in which A_i denotes the cross-sectional area for the ith element and $\underline{A}_i$ and $\bar{A}_i$ its lower and upper bounds, respectively. The generalized variable M_k denotes the material constants for material k, and k_M is the number of possible materials. The constants c_i are scaling factors that multiply the corresponding weight densities and take into account the relative cost of the different materials. From a physical point of view, the constants c_i have the units of cost per unit of weight and in this paper are used as relative costs of materials. The term n is the number of independent structural elements to be designed, and m is the total number of behavior constraints.

Equation (1a) is then the cost of the weight of the structure when a given set of materials M_k and sizing variables A_i are chosen for the structural elements. If all of the constants c_k are set to 1, the problem stated by Eqs. (1) reduces to a traditional weight minimization problem. The behavior constraints g_j from Eq. (1c) include static displacement constraints, stress constraints, and natural frequency constraints.

Modeling

The (0, 1) formulation of the material selection problem makes use of decision variables α_{ik} that indicate the material to be chosen for the ith element. That is, α_{ik} is 1 if material k is selected and 0 otherwise. For a mathematical model, it is considered that each structural member is composed of a series of members all with the same area, acting in parallel, each one corresponding to a different candidate material. For a structural model, following the notation introduced in Ref. 6, these combined elements will generate a generalized structural element denoted hybrid element. The hybrid element properties are obtained by adding the properties given by each of the elements in parallel, multiplied by the corresponding variables α_{ik}. The equivalent mechanical stiffness for a hybrid element is given by

$$k_i = \sum_{k=1}^{k_M} \alpha_{ik} k_{ik} \tag{2}$$

in which k_{ik} denotes the stiffness for element i when material k is used,

$$k_{ik} = \frac{E_k A_i}{L_i} \tag{3}$$

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where E_k is the Young's modulus for material k, and A_i, and L_i are the area and length of element i, respectively. Introducing Eq. (3) into Eq. (2),

$$k_i = \left(\sum_{k=1}^{k_M} \alpha_{ik} E_k\right) \frac{A_i}{L_i} \tag{4}$$

Also the equivalent mass of the hybrid element is given by

$$m_i = \left(\sum_{k=1}^{k_M} \alpha_{ik} \rho_k\right) A_i L_i \tag{5}$$

where ρ_k is the mass density for material k.

Since the elements are in parallel, the force in the hybrid element is given by

$$F_i = \sum_{k=1}^{k_M} \alpha_{ik} F_{ik} \tag{6}$$

in which F_{ik} denotes the force generated in element i when material k is used. For an elastic material,

$$F_{ik} = k_{ik} \delta_i \tag{7}$$

or substituting k_{ik} from Eq. (3),

$$F_{ik} = \frac{E_k A_i}{L_i} \delta_i \tag{8}$$

where δ_i denotes the elastic change in length in element i.

Substituting Eq. (8) into Eq. (6), one can write the force for the hybrid element as

$$F_i = \left(\sum_{k=1}^{k_M} \alpha_{ik} E_k\right) \frac{A_i \delta_i}{L_i} \tag{9}$$

The forces F_{ik} can be easily recovered from F_i by

$$F_{ik} = \frac{\alpha_{ik} k_{ik}}{k_i} F_i \tag{10}$$

or

$$F_{ik} = \frac{\alpha_{ik} E_k}{\sum_{r=1}^{k_M} \alpha_{ir} E_r} F_i \tag{11}$$

From Eq. (11) the stress for element i when material k is used is given by

$$\sigma_{ik} = \left(\frac{\alpha_{ik} E_k}{\sum_{r=1}^{k_M} \alpha_{ir} E_r}\right) \frac{F_i}{A_i} \tag{12}$$

Since stress limits must be satisfied for each possible material selected, the stress constraints for element i can be written as

$$\sigma_k^L \le \sigma_{ik} \le \sigma_k^U \qquad k = 1, \ldots, k_M \tag{13}$$

Equations (13) indicate that there are k_M constraints for each member i. This number of constraints can be large depending on the number of possible materials to choose from, imposing additional computational burden for the optimizer. A more convenient approach is to condense the k_M stress constraints for element i into a single stress constraint that considers all possible materials.

The variables α_{ik} are (0, 1) variables that indicate if material k is selected (1) or not selected (0). Since only one material must be selected for each member, the following constraints should be also considered:

$$\sum_{k=1}^{k_M} \alpha_{ik} = 1 \qquad i = 1, \ldots, n \tag{14}$$

In a practical implementation the equality constraints given by Eq. (14) are not explicitly considered, but rather the number of discrete variables is reduced using these relations.

Considering that Eq. (14) is satisfied, the set of constraints given by Eqs. (13) can be written as a single constraint of the form

$$\sum_{k=1}^{k_M} \alpha_{ik} \sigma_k^L \le \sum_{k=1}^{k_M} \alpha_{ik} \sigma_{ik} \le \sum_{k=1}^{k_M} \alpha_{ik} \sigma_k^U \tag{15}$$

The equivalence of the constraints given by Eqs. (13) and (15) is easily proven since at the optimum (which is discrete) only one of the α_{ik} will be 1 and the rest 0. Finally, replacing the expression for σ_{ik} given from Eq. (12) into Eq. (15), the stress constraint for element i is given by

$$\sum_{k=1}^{k_M} \alpha_{ik} \sigma_k^L \le \frac{\sum_{k=1}^{k_M} \alpha_{ik}^2 E_k}{\sum_{k=1}^{k_M} \alpha_{ik} E_k} \frac{F_i}{A_i} \le \sum_{k=1}^{k_M} \alpha_{ik} \sigma_k^U \tag{16}$$

Denoting the stress in element i by

$$\sigma_i = \frac{F_i}{A_i} \tag{17}$$

the single stress constraint is then given by

$$\sigma_i^L \le \sigma_i \le \sigma_i^U \tag{18}$$

where

$$\sigma_i^L = \frac{\sum_{k=1}^{k_M} \alpha_{ik} E_k \sum_{k=1}^{k_M} \alpha_{ik} \sigma_k^L}{\sum_{k=1}^{k_M} \alpha_{ik}^2 E_k} \tag{19a}$$

$$\sigma_i^U = \frac{\sum_{k=1}^{k_M} \alpha_{ik} E_k \sum_{k=1}^{k_M} \alpha_{ik} \sigma_k^U}{\sum_{k=1}^{k_M} \alpha_{ik}^2 E_k} \tag{19b}$$

Thus, the stress constraints for element i can be replaced by a single constraint on σ_i in which upper and lower limits (σ_i^L, σ_i^U) are weighted quantities that depend explicitly on the (0, 1) design variables α_{ik}.

Problem Formulation in Terms of (0, 1) Variables

Using the (0, 1) variables α_{ik} introduced in the previous section, the optimization problem given by Eqs. (1) takes the following form:

Minimize

$$\sum_{i=1}^{n} \sum_{k=1}^{k_M} \alpha_{ik} c_k \rho_k A_i L_i \tag{20a}$$

subject to

$$g_j(A, \alpha) \le 0 \qquad j = 1, \ldots, m \tag{20b}$$

$$\underline{A}_i \le A_i \le \bar{A}_i \qquad i = 1, \ldots, n \tag{20c}$$

$$\sum_{k=1}^{k_M} \alpha_{ik} = 1 \qquad i = 1, \ldots, n \tag{20d}$$

$$\alpha_{ik} = 0 \text{ or } 1 \qquad i = 1, \ldots, n, k = 1, \ldots, k_M \tag{20e}$$

in which the vectors A and α represent the continuous sizing variables and the (0, 1) material selection variables, respectively.

The problem represented by Eqs. (20) is highly nonlinear, and the constraints g_j are implicit functions of the design variables. Therefore, direct application of a nonlinear programming algorithm for its solution is computationally impractical even for small systems. A more tractable approach is to replace the solution of this implicit nonlinear problem by the solution of a sequence of explicit approximate problems in which only the critical and potentially critical constraints are retained and move limits are used to protect the approximations. In this context, the presence of (0, 1) variables induces nondifferentiability of the various functions with respect to these variables. This difficulty is overcome by the use of branch and bound techniques, where, at each stage, the discrete variables remain continuous within [0, 1] bounds. This allows the same treatment of discrete and continuous variables.

Branch and bound strategies consist of solving a tree of continuous relaxations of the original discrete problem until a feasible discrete solution is achieved. The vertices of the tree correspond to a problem where a subset of discrete variables are held fixed at their bounds and the remaining (0, 1) variables are allowed to be continuous between 0 and 1. These vertices are arranged into levels in the following way; there is a single vertex at level 0 (called the root of the tree) that corresponds to the complete continuous relaxation of the original problem. Subsequent levels are generated by choosing a discrete variable α_{ik} and generating two vertices for which α_{ik} has the value 0 for the first node and 1 for the second node. Therefore, a partition of the discrete set is generated where only a subset of the discrete variables is relaxed (i.e., allowed to be continuous). For a more detailed discussion of the branch and bound algorithm, see Ref. 7.

Problem Solution

A problem similar in structure to the one formulated here was solved in Ref. 8, in which the (0, 2) variables represented discrete location variables for actuators and sensors in actively controlled structures. The continuous variables were sizing variables for structural members and feedback gains for the control system. The solution procedure introduced in Ref. 8 is a simplification of the classical branch and bound strategy. In this approach, the root of the enumeration tree (continuous relaxation) was solved by generating and solving a sequence of approximate problems. During this phase all (0, 1) variables are treated as continuous variables and bounded such that $0 \leq \alpha_{ik} \leq 1$. When the continuous solution is obtained, all those (0, 1) variables that seek their lower or upper bounds are rounded off to zero or one and eliminated as design variables, thus reducing the number of discrete variables. Starting from the continuous design, in which the number of (0, 1) variables has been reduced, a new sequence of approximate problems is generated and solved using branch and bound, obtaining, therefore, a discrete solution. The advantage of the procedure resides in the fact that the number of discrete variables is reduced and thus the dimension of the enumeration tree is smaller even though the true discrete solution may not be reached because of the rounding off of certain discrete variables. The other main disadvantage is that since there is no feasible discrete solution available, the branch and bound algorithm may require the enumeration of large portions of the tree before some nodes can be fathomed (eliminated from further search). A discrete feasible solution provides an upper bound for the optimal discrete solution of the problem. When this bound is available, a node that has an objective higher than the upper bound can be fathomed, thus also eliminating all of its descendents from further search.

In this paper, a different approach is investigated. Since the efficiency of branch and bound depends on the ability to fathom at early stages those nodes that do not contain the optimal solution, it is crucial to have a good feasible discrete solution at the beginning of the enumeration of the tree. For this purpose an auxiliary continuous problem is solved using a sequence of approximate problems (phase I). The solution obtained is used to generate a feasible discrete solution. Then a formal branch and bound approach is used to determine the final solution (phase II). This discrete phase is implemented by solving a new sequence of approximate problems, each one of which is solved using branch and bound.

Phase I: Continuous Problem and Initial Discrete Feasible Design

In this first phase, a modified continuous problem is solved to obtain a feasible discrete solution.

Consider the auxiliary continuous problem:

Minimize

$$\sum_{i=1}^{n} \bar{c}_i \bar{\rho}_i A_i L_i \tag{21a}$$

subject to

$$q_j^L \leq q_j(\bar{\boldsymbol{E}}, \boldsymbol{A}) \leq q_j^U \qquad j \in J \tag{21b}$$

$$\tilde{\sigma}_i^L \leq \sigma_i(\bar{\boldsymbol{E}}, \boldsymbol{A}) \leq \tilde{\sigma}_i^U \qquad i = 1, \ldots, m \tag{21c}$$

$$\tilde{\omega}_k^L \leq \omega_k(\bar{\boldsymbol{E}}, \boldsymbol{A}) \leq \tilde{\omega}_k^U \qquad k \in K \tag{21d}$$

$$\tilde{A}_i^L \leq A_i \leq \tilde{A}_i^U \qquad i = 1, \ldots, n \tag{21e}$$

For this problem the vectors of parameters $\bar{\boldsymbol{c}}$, $\bar{\boldsymbol{\rho}}$, and $\bar{\boldsymbol{E}}$ are nominal properties for each element. The limits on displacements (q_j^L, q_j^U) are the actual bounds for the problem, whereas the limits on stresses $(\tilde{\sigma}_i^L, \tilde{\sigma}_i^U)$, frequencies $(\tilde{\omega}_k^L, \tilde{\omega}_k^U)$, and areas $(\tilde{A}_i^L, \tilde{A}_i^U)$ are to be determined such that the continuous solution allows finding a feasible discrete solution.

The solution for problem (21) is denoted by A_{i0}, $i = 1, \ldots, n$. Then, for a given member, the stiffness and mass properties are given by the quantities (index i omitted)

$$k_o = \frac{\bar{E} A_0}{L} \tag{22a}$$

$$m_o = \bar{\rho} A_0 L \tag{22b}$$

If material j is chosen for this element, and the corresponding area is A, we have

$$k_j = \frac{E_j A}{L} \tag{23a}$$

$$m_j = \rho_i A L \tag{23b}$$

Since problem (21) is feasible with respect to displacements, we can maintain this feasibility by preserving the stiffness, i.e., $k_0 = k_j$ with this choice the area A required is given by

$$A = \frac{\bar{E} A_0}{E_j} \tag{24}$$

Now, since for this area the stress constraints may not be satisfied, the design can be scaled to the constraint boundary. The scaled stress is then given by

$$\sigma = \frac{F_0}{\mu A} = \frac{F_0 E_j}{\mu \bar{E} A_0} = \frac{\sigma_0 E_j}{\mu \bar{E}} \tag{25}$$

where F_0 is the axial force, which does not change when scaling, and μ is the scaling factor. Imposing the actual stress constraints for material j, the following relation must hold,

$$\sigma_j^L \leq \frac{\sigma_0 E_j}{\mu \bar{E}} \leq \sigma_j^U \tag{26}$$

from where we can obtain the required scaling parameter

$$\sigma_0 \geq 0 \Rightarrow \mu \geq \frac{\sigma_0 E_j}{\bar{E} \sigma_j^U} \tag{27a}$$

$$\sigma_0 < 0 \Rightarrow \mu \geq \frac{\sigma_0 E_j}{\bar{E} \sigma_j^L} \tag{27b}$$

the scaling parameter is also limited by the actual bounds for the element cross-sectional area; therefore, from Eq. (24) we have

$$\underline{A} \leq \frac{\mu \bar{E} A_0}{E_j} \leq \bar{A} \tag{28}$$

or

$$\frac{\underline{A} E_j}{\bar{E} A_0} \leq \mu \leq \frac{\bar{A} E_j}{\bar{E} A_0} \tag{29}$$

Also, since the displacement constraints must be satisfied, the stiffness must not decrease, and therefore a third condition for μ is given by

$$\mu \geq 1 \tag{30}$$

Then, the scaling parameter must satisfy Eqs. (28), (29), and (30). For this system of inequalities to admit a solution, the following conditions must be satisfied:

$$A_0 \leq \frac{\bar{A} E_j}{\bar{E}} \tag{31a}$$

$$\sigma_0 \leq \frac{\bar{A}\sigma_j^U}{A_0} \quad \text{for} \quad \sigma_0 \geq 0 \tag{31b}$$

$$\sigma_0 \geq \frac{\bar{A}\sigma_j^L}{A_0} \quad \text{for} \quad \sigma_0 < 0 \tag{31c}$$

Equations (31) give the basis for defining stress and area bounds for problem (21);

$$\tilde{A}^L = \underline{A} \tag{32a}$$

$$\tilde{A}^U = \frac{\bar{A}}{\bar{E}} \operatorname*{Min}_j E_j \tag{32b}$$

$$\tilde{\sigma}^U = \frac{\bar{E}\,\mathrm{Min}_j\, \sigma_j^U}{\mathrm{Min}_j\, E_j} \tag{32c}$$

$$\tilde{\sigma}^L = \frac{\bar{E}\mathrm{Max}_j \sigma_j^L}{\mathrm{Min}_j E_j} \tag{32d}$$

Note that the limits given by Eqs. (32b–32d) only apply if $\bar{A} < +\infty$.

For the case of frequency constraints, the ratio for the mass of the element when material j is used as compared with the resulting mass given by the optimal solution of problem (21) is given by

$$\frac{m_0}{m_j} = \frac{\bar{\rho} E_j}{\bar{E}\rho_j} \tag{33}$$

defining

$$\beta_j^2 = \frac{\bar{\rho} E_j}{\bar{E}\rho_j} \tag{34}$$

then

$$m_0 = \beta_j^2 m_j \tag{35}$$

and the bounds for the actual frequencies are then given by

$$\omega_{\max} = \omega_0 \operatorname*{Max}_j \beta_j \tag{36a}$$

$$\omega_{\min} = \omega_0 \operatorname*{Min}_j \beta_j \tag{36b}$$

from where it is seen that the bounds for ω have to be chosen as

$$\tilde{\omega}^L = \omega^L \sqrt{\frac{\bar{E}}{\bar{\rho}}} \operatorname*{Max}_j \sqrt{\frac{\rho_j}{E_j}} \tag{37a}$$

$$\tilde{\omega}^U = \omega^U \sqrt{\frac{\bar{E}}{\bar{\rho}}} \operatorname*{Min}_j \sqrt{\frac{\rho_j}{E_j}} \tag{37b}$$

Now from Eqs. (27), (29), and (30) the minimum scaling parameter required for material j is given by

$$\mu_j = \mathrm{Max}\left\{1, \frac{AE_j}{\bar{E}A_0}, \frac{\sigma_0 E_j}{\bar{E}\sigma_j^U} \text{ if } \sigma_0 \geq 0, \frac{\sigma_0 E_j}{\bar{E}\sigma_j^L} \text{ if } \sigma_0 < 0\right\} \tag{38}$$

and then material j is selected if it gives the minimum weight among all possible materials for the element, i.e.,

$$j = \mathrm{Arg\,min}\left\{\frac{\mu_i \rho_i c_i}{E_j}\right\} \tag{39}$$

If all of the c_i are 1, the rule given by Eq. (39) selects the material for minimum weight.

Approximate Problem Generation

In seeking robust approximations of behavior constraints it is important to appreciate the flexibility offered by the use of intermediate design variables and intermediate response quantities. These ideas originally introduced in Ref. 2 have been recently applied with considerable success to approximations for natural frequencies,[9] static stress,[10] complex eigenvalues,[11] steady-state dynamic displacements,[12] and transient peak dynamic displacements.[13]

For the successful implementation of the idea of intermediate response quantities, it is essential to find a set of appropriate intermediate design variables. These intermediate design variables should be chosen such that first-order approximations of the intermediate response quantities are of high quality with respect to these intermediate design variables.

In this work, the actual design variables are the areas A_i for structural members and the decision variable α_{ik}. An examination of Eqs. (4) and (5) reveals that the key matrices involved in both static and eigenvalue analyses (i.e., $[M]$ and $[K]$) are linear functions of the following variables:

$$X_{1i} = \sum_{k=1}^{k_M} \alpha_{ik} E_k A_i \tag{40a}$$

$$X_{2i} = \sum_{k=1}^{k_M} \alpha_{ik} \rho_k A_i \tag{40b}$$

These variables are chosen in this work as intermediate design variables. For stress constraints, the hybrid element forces F_i are used as intermediate response quantities,[10] and the approximate stress constraints are constructed using Eq. (17) by replacing F_i by its approximation $\tilde{F}_i$ in terms of X_{1i} and X_{2i}. For natural frequency constraints the modal energies U_n and T_n (Ref. 9) are approximated in terms of X_{1i} and X_{2i} using first-order information.

It is interesting to note that by using intermediate response quantities and intermediate design variables the sensitivities required to construct the approximations do not depend on the number of possible materials. In fact, the effort necessary to construct the approximate problems is identical to the case of a pure sizing problem in which the only design variables are A_i. The number of (0, 1) variables only affects the solution of each approximate problem, and if this number is large, the combinatorial nature of the problem will be reflected in the time needed to optimize the approximate problem, which is explicit.

Numerical Examples

The synthesis methodology described previously has been implemented in a research computer program that is operational on the IBM ES/9000 Model 900 computer at UCLA. The continuous optimization phase was implemented using DOT (Ref. 14) and the branch and bound phase using routines from DOC (Ref. 15).

Problem 1: 72-Member Truss

A 72-bar space truss structure has been chosen to test and verify the method presented in this paper. Figure 1 shows the geometry of the structure and the node and the member numbering system in detail for the uppermost tier. Two independent static load conditions shown in Table 1 are considered for this problem. The symmetry of the structure and the loading conditions are such that the number of independent sizing design variables can be reduced to 16 using design variable linking. Four possible materials are considered for

Table 1 Load conditions, 72-bar example problem

Load condition	Node	Load, lb P_x	P_y	P_z
1	1	5000	5000	−5000
2	1	0	0	−5000
	2	0	0	−5000
	3	0	0	−5000
	4	0	0	−5000

Table 2 Material properties, 72-bar example problem

Material	ρ, pci	E, psi	σ^a, psi	c, \$/lb
High-tension steel	0.284	0.298×10^8	$\pm 0.45 \times 10^5$	1.0
Aluminum	0.100	0.100×10^8	$\pm 0.25 \times 10^5$	2.0
Titanium	0.165	0.170×10^8	$\pm 0.81 \times 10^5$	2.5
Magnesium	0.060	0.060×10^8	$\pm 0.19 \times 10^5$	2.2

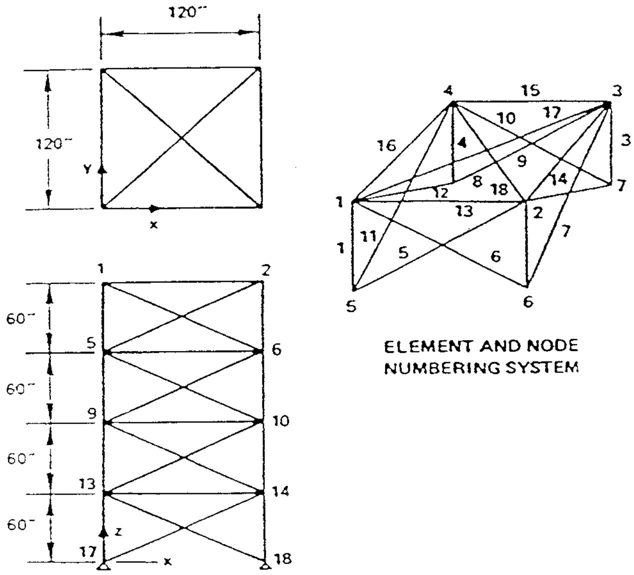

Fig. 1 72-member truss.

each independent design group. The properties of these four materials are given in Table 2.

Move limits of 80% are used throughout this example. All initial cross-sectional areas are 1.0 in.^{2} and initial α are 0.5 for all cases. The convergence criterion for the continuous phase is that the difference between the objective functions for three consecutive iterations is less or equal to 1%. The convergence criterion for the discrete phase is 0.1%.

Case 1: Weight Minimization

In this problem the objective function to be minimized is the total weight of the structure. The displacements of nodes 1–4 are limited to ± 0.25 in. in the x and y directions for both load conditions. The stresses in each member are also constrained, and the allowable values are given in Table 2. Side constraints are imposed on the areas of the members such that $A_i \geq 0.1$ in.^{2}.

Eight runs were made for this problem. The first four runs are used for comparison purposes and do not involve material selection. In these runs all members are assumed to be steel, aluminum, titanium, and magnesium, respectively. In the fifth and sixth runs the weight of the structure is minimized, considering only steel and aluminum as candidate materials. This problem has been solved in Ref. 4 and is used for comparison purposes. For the fifth run the problem was solved using the scheme discussed in this paper, and for the sixth run the method presented in Ref. 8 was used. Finally, the seventh and eighth runs involve material selection with all four possible

Table 3 Optimal designs, single material weight minimization, problem 1

Element	Area, in.^{2} Steel	Aluminum	Titanium	Magnesium
1–4	0.1000	0.1559	0.1000	0.1948
5–12	0.1875	0.5451	0.3181	0.8952
13–16	0.1305	0.4093	0.2331	0.6899
17–18	0.1866	0.5612	0.3237	0.9125
19–22	0.1734	0.5259	0.3002	0.8728
23–30	0.1763	0.5154	0.3043	0.8441
31–34	0.1000	0.1000	0.1000	0.1000
35–36	0.1000	0.1110	0.1000	0.1797
37–40	0.4285	1.2653	0.7504	2.0884
41–48	0.1742	0.5096	0.3012	0.8364
49–52	0.1000	0.1001	0.1000	0.1000
53–54	0.1000	0.1000	0.1000	0.1000
55–58	0.6483	1.8834	1.1190	3.1046
59–66	0.1750	0.5092	0.3021	0.8347
67–70	0.1001	0.1001	0.1000	0.1000
71–72	0.1000	0.1000	0.1000	0.1000
Weight, lb	362.38	378.75	384.10	364.60
Cost, \$	362.38	757.50	960.25	802.12

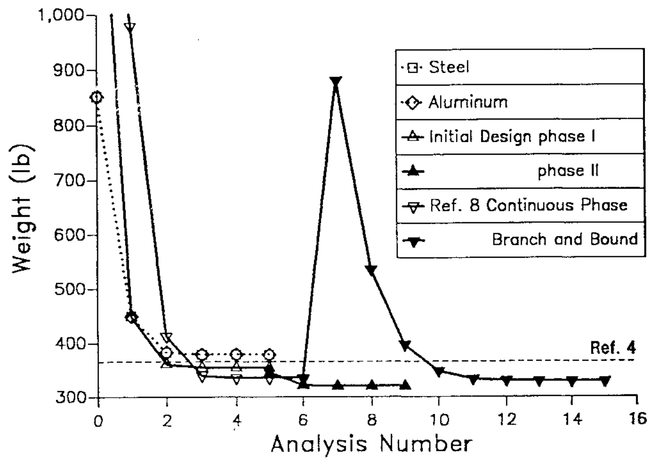

Fig. 2 Iteration histories, weight minimization, problem 1.

materials for each member solved using the present scheme and the method of Ref. 8, respectively.

The final designs for the first four runs are given in Table 3. The final designs for runs five through eight along with the solutions obtained in Ref. 4 are given in Table 4. The iteration histories are plotted in Fig. 2. For the first four runs the active constraints at the optimum are the displacements for node 1 in the x and y directions for load condition 1 and the stress constraints for elements 1–4 for the second run (all aluminum) and the fourth run (all magnesium). The optimal weights and associated costs are given in Table 3. For the fifth run (steel-aluminum) the final weight is 319.86 lb, which is smaller than the best single material optimization (all steel) by 12% but the associated cost of \$404.61 is 12% higher. The active constraints for the final design are the same as in the second run. From Table 4 it is also seen that the material selection procedure presented in this paper gives the same selection as the one given in Ref. 4 with the exception that the weight is 13% lower. Probably this difference in weight is due to the convergence criterion used in Ref. 4. The sixth run (method from Ref. 8) gives an optimal weight of 329.17 lb, which is still 10% lower than Ref. 4 but with a different material selection. The iteration histories in Fig. 2 show the different performance of the two methods. The initial continuous design phase (after scaling) gives a discrete solution that is very similar in weight to the optimal discrete solution. On the other hand, by using the method of Ref. 8, after roundoff, the weight increases until a discrete feasible solution is obtained, and then the objective decreases monotonically until the optimum is obtained. Finally, when all four possible materials are considered as candidates, the final weight is 306.70 lb and the associated cost is \$388.24. With minor

Table 4 Optimal designs, material selections weight minimization, problem 1

Element	S, A,[a] present method	S, A,[a] method of Ref. 8	Ref. 4	S, A, T, M,[a] present method	S, A, T, M,[a] method of Ref. 8
	Area, in.^{2}				
1–4	0.1565 (A)	0.1560 (A)	0.1693 (A)	0.1969 (M)	0.1971 (M)
5–12	0.1835 (S)	0.1893 (S)	0.1850 (S)	0.1823 (S)	0.1822 (S)
13–16	0.1365 (S)	0.1377 (S)	0.1211 (S)	0.1375 (S)	0.1375 (S)
17–18	0.1885 (S)	0.1949 (S)	0.1741 (S)	0.1858 (S)	0.1860 (S)
19–22	0.1742 (S)	0.4677 (A)	0.1768 (S)	0.1741 (S)	0.1739 (S)
23–30	0.1735 (S)	0.1786 (S)	0.1770 (S)	0.1688 (S)	0.1689 (S)
31–34	0.1000 (A)	0.1000 (A)	0.1053 (A)	0.1000 (M)	0.1000 (M)
35–36	0.1002 (A)	0.1000 (A)	0.1053 (A)	0.1448 (M)	0.1500 (M)
37–40	0.4244 (S)	0.4325 (A)	0.4348 (S)	0.4208 (S)	0.4207 (S)
41–48	0.1716 (S)	0.1720 (S)	0.1752 (S)	0.1644 (S)	0.1645 (S)
49–52	0.1000 (A)	0.1000 (A)	0.1053 (A)	0.1000 (M)	0.1000 (M)
53–54	0.1000 (A)	0.1000 (A)	0.1053 (A)	0.1000 (M)	0.1000 (M)
55–58	0.6324 (S)	1.7268 (A)	0.6458 (S)	0.6215 (S)	0.6217 (S)
59–66	0.1714 (S)	0.1726 (S)	0.1755 (S)	0.1750 (S)	0.1751 (S)
67–70	0.1000 (A)	0.1000 (A)	0.1053 (A)	0.1000 (M)	0.1000 (S)
71–72	0.1000 (A)	0.1000 (A)	0.1053 (A)	0.1000 (M)	0.1000 (S)
Weight, lb	319.86	329.17	366.22	306.70	307.12
Cost, $	404.61	541.73	504.75	388.24	389.03

[a]Candidate materials: steel (S), aluminum (A), titanium (T), and magnesium (M).

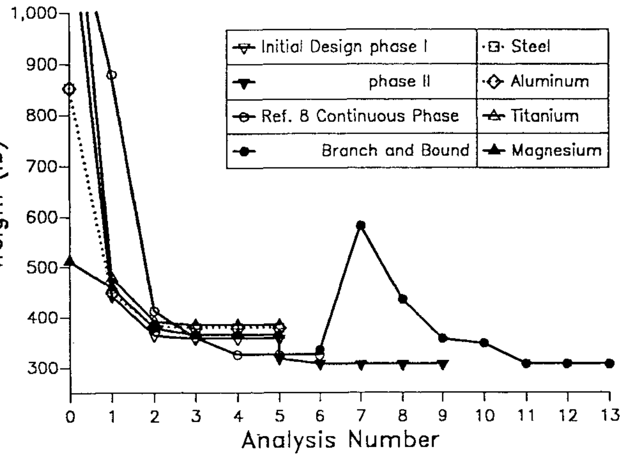

Fig. 3 Iteration histories, cost minimization, problem 1.

differences between the solutions obtained by the two methods, the iteration histories have a similar behavior as for the previous case.

Case 2: Cost Minimization

In this second case the objective function is the total cost of the structure, whereas the constraints are the same as those in case 1. The optimal solution for the single material case is the same as in the weight minimization case, and the optimal cost is given in Table 3.

Three runs were made for this case. The first run considers only steel and aluminum as candidate materials and is used to compare with the solution of Ref. 8. The second and third runs consider all four possible materials and are solved using the scheme presented in this paper and the method of Ref. 8. The final designs for these three runs along with the solution given in Ref. 4 are given in Table 5. The iteration histories are plotted in Fig. 3. As expected, all three results give a pure steel design. The iteration histories in this case have a different behavior as compared with the weight minimization case. Since the selection is trivial, the continuous screening solution (method of Ref. 8) is almost discrete, and the objective function does not change significantly during the branch and bound phase.

Problem 2: Buckling Constraints

In this second problem additional buckling constraints are imposed for all members. To use the area of each element as the only design variable, it is assumed that the truss members are thin hollow tubes with mean diameter D and thickness t. The allowable

Table 5 Optimal designs, material selection cost minimization, problem 1

Element	S, A,[a] present method	Ref. 4	S, A, T, M,[a] present method	S, A, T, M,[a] method of Ref. 8
	Area, in.^{2}			
1–4	0.1000 (S)	0.1019 (S)	0.1000 (S)	0.1000 (S)
5–12	0.1848 (S)	0.1930 (S)	0.1837 (S)	0.1874 (S)
13–16	0.1282 (S)	0.1191 (S)	0.1345 (S)	0.1304 (S)
17–18	0.1848 (S)	0.1852 (S)	0.1886 (S)	0.1886 (S)
19–22	0.1711 (S)	0.1758 (S)	0.1661 (S)	0.1730 (S)
23–30	0.1735 (S)	0.1772 (S)	0.1814 (S)	0.1792 (S)
31–34	0.1000 (S)	0.1019 (S)	0.1000 (S)	0.1000 (S)
35–36	0.1001 (S)	0.1019 (S)	0.1002 (S)	0.1000 (S)
37–40	0.4378 (S)	0.4300 (S)	0.4272 (S)	0.4280 (S)
41–48	0.1718 (S)	0.1755 (S)	0.1746 (S)	0.1742 (S)
49–52	0.1000 (S)	0.1019 (S)	0.1001 (S)	0.1000 (S)
53–54	0.1000 (S)	0.1019 (S)	0.1000 (S)	0.1000 (S)
55–58	0.7013 (S)	0.6473 (S)	0.6530 (S)	0.6492 (S)
59–66	0.1725 (S)	0.1762 (S)	0.1723 (S)	0.1745 (S)
67–70	0.1000 (S)	0.1019 (S)	0.1001 (S)	0.1000 (S)
71–72	0.1000 (S)	0.1019 (S)	0.1000 (S)	0.1000 (S)
Weight (lb)	362.74	417.63	362.57	362.42
Cost ($)	362.74	417.63	362.57	362.42

[a]Candidate materials: steel (S), aluminum (A), titanium (T), and magnesium (M).

compressive stresses are given as the most critical between the following:

Euler buckling:

$$\sigma_e = \frac{\pi^2 E D^2}{8L^2}$$

Local buckling:

$$\sigma_c = \frac{0.4Et}{D} = \frac{0.4EA}{\pi D^2}$$

The mean diameter is assumed fixed to $D = 2.5$ in. The thin tube requirement is expressed as an upper bound on the area:

$$\frac{D}{t} \geq 10 \Rightarrow A \leq \frac{\pi D^2}{10}$$

Table 6 Optimal designs, material selection weight minimization, problem 2

	Area, in.^{2}				
Element	Steel	Aluminum	Titanium	Magnesium	Selection
1–4	0.1001	0.1881	0.1037	0.3166	0.1154 (T)
5–12	0.2205	0.6279	0.4005	1.0204	0.1926 (S)
13–16	0.1384	0.4211	0.2402	0.7870	0.1508 (S)
17–18	0.2223	0.7019	0.3743	1.1846	0.2528 (S)
19–22	0.1685	0.5060	0.2892	0.9605	0.1776 (S)
23–30	0.1608	0.4958	0.2888	0.9586	0.1699 (S)
31–34	0.1001	0.1000	0.1000	0.1000	0.1006 (M)
35–36	0.1000	0.1000	0.1000	0.1782	0.1000 (M)
37–40	0.4037	1.2153	0.7050	1.9630	0.4194 (S)
41–48	0.1722	0.4929	0.2847	0.9624	0.1627 (S)
49–52	0.1000	0.1000	0.1000	0.1000	0.1002 (M)
53–54	0.1000	0.1000	0.1000	0.1000	0.1001 (M)
55–58	0.6156	1.8302	1.0576	1.9630	0.6211 (S)
59–66	0.1685	0.4938	0.2854	0.9643	0.1717 (S)
67–70	0.1000	0.1001	0.1000	0.1000	0.1000 (M)
71–72	0.1000	0.1000	0.1000	0.1000	0.1000 (M)
Weight, lb	417.99	384.88	388.94	389.56	359.69

Table 7 Optimal designs, material selection weight minimization, problem 3

	Area, in.^{2}				
Element	Steel	Aluminum	Titanium	Magnesium	Selection
1–4	0.1000	0.1921	0.1068	0.3307	0.1156(T)
5–12	0.2291	0.6470	0.4111	1.0041	0.1929 (S)
13–16	0.1000	0.2962	0.1665	0.5845	0.1476 (S)
17–18	0.2351	0.7503	0.4032	1.3262	0.2524 (S)
19–22	0.1051	0.3315	0.1781	0.6882	0.1715 (S)
23–30	0.1532	0.4652	0.2761	0.8686	0.1767 (S)
31–34	0.1000	0.1000	0.1000	0.1000	0.1004 (M)
35–36	0.1000	0.1000	0.1000	0.1000	0.1006 (M)
37–40	0.2923	0.9547	0.5272	1.9586	0.4297 (S)
41–48	0.2300	0.6078	0.3712	1.0591	0.1717 (S)
49–52	0.1000	0.1000	0.1000	0.1000	0.1005 (M)
53–54	0.1000	0.1000	0.1000	0.1000	0.1001 (M)
55–58	0.4926	1.5761	0.8887	1.9630	0.5617 (S)
59–66	0.2868	0.7664	0.4527	1.2248	0.1710 (S)
67–70	0.1000	0.1001	0.1000	0.1000	0.1001 (M)
71–77	0.1000	0.1000	0.1000	0.1000	0.1000 (M)
Weight, lb	447.67	404.44	410.98	397.41	360.20

In this problem only the weight minimization case is considered. The optimal designs for the case of single material and material selection [steel, aluminum, titanium, and magnesium (S, A, T, M)] are given in Table 6. The iteration histories are plotted in Fig. 4. The displacement constraints for node 1 in the x and y directions and the stress constraints for elements 6, 12, and 17 for load condition 1 are active for the optimal solution for all five runs. With the exception of the first run (all steel), the stress constraints for elements 1–4 for load condition 2 are also active at the final solution. From Table 6 it is seen that the final weight when material selection is allowed is 359.69 lb, which is 6.5% smaller than the best single material case (aluminum with 384.94 lb).

Problem 3: Frequency Constraints

Finally, the third example problem involves the weight minimization of the structure with the same constraints as in problem 2 with the addition of frequency constraints given by $\omega_{1,2} \geq 35$ Hz (first mode is repeated) and $\omega_3 \geq 60$ Hz.

The optimal designs are given in Table 7, and the iteration histories are plotted in Fig. 5. The active constraints for all five runs at the final solution are the same as in problem 2 in addition to the frequency constraints. In this case the material selected also has the same pattern as in problem 2 with a final weight of 360.20 lb, which is 12% lower than the best single material design (all aluminum with 410.98 lb).

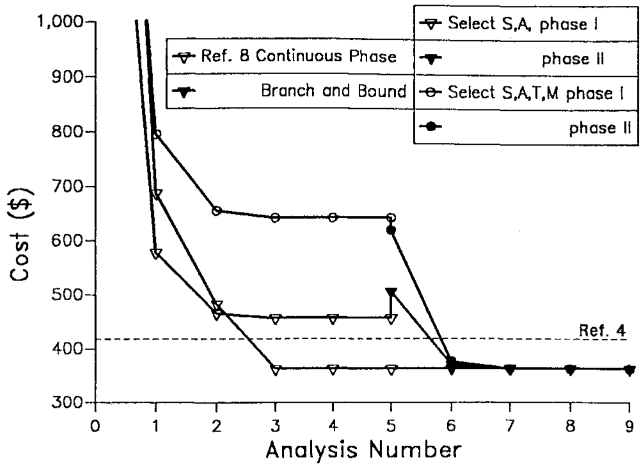

Fig. 4 Iteration histories, problem 2.

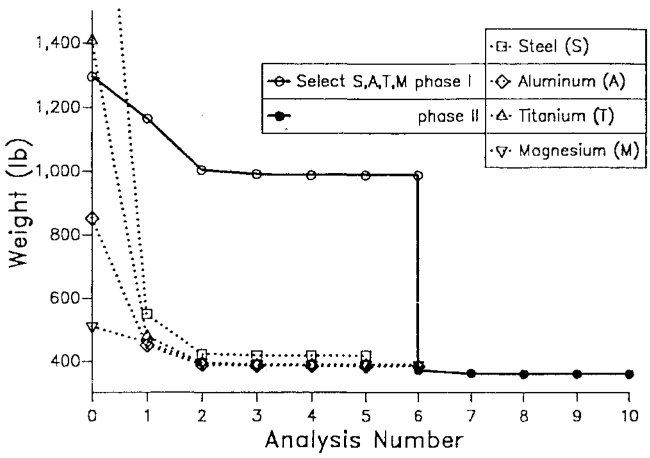

Fig. 5 Iteration histories, problem 3.

Conclusions

A synthesis methodology for optimal material selection for structures modeled as an assemblage of truss elements has been presented. Areas for structural elements and (0, 1) material decision variables are used as independent design variables for the synthesis problem. The design problem is posed as a general mixed (0, 1) continuous programming problem. The objective functions considered are the total weight of the structure or the cost of the material. Constraints are imposed on static displacements, static stresses, and natural frequencies. The mixed (0, 1) continuous design problem is solved by combining approximation concepts and branch and bound techniques. High-quality approximations are constructed using intermediate response quantities and intermediate design variables.

The numerical examples demonstrate the feasibility of the proposed design technique. Near-optimal solutions can typically be obtained within 5–10 design iterations. Branch and bound techniques combined with approximation concepts prove to be efficient in terms of the number of analyses required for convergence. The continuous initialization phase (phase I) proved to give good initial feasible discrete designs for the discrete branch and bound phase.

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